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The flow due to a slender ship moving over a wavy wall in shallow water

A. PLOTKIN

Department of Aerospace Engineering, University of Maryland, College Park, Maryland, U.S.A.

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SUMMARY
The problem under investigation is the unsteady subcritical potential flow generated by a slender ship translating over a wavy wall in shallow water. The method of matched asymptotic expansions is used to take advantage of the simplified governing equations in the near and far fields. The vertical force and pitching moment coefficients are calculated as functions of the reduced frequency and Froude number with a view towards possible application to ship safety considerations.

1. Introduction

For primarily safety-related considerations, there has been much recent interest in the problem of ship motions in restricted waters. This has given rise to a number of studies of shallow-water potential flows. The method of matched asymptotic expansions has proven to be invaluable in these analytical efforts. It first appeared in this role in the literature in a paper by Tuck [1] which analyzed the steady longitudinal flow past a slender ship in constant-depth shallow water. Slender-body theory was used to describe the flow field in the inner region near the ship and shallow-water theory provided the outer region description. Other steady constant-depth solutions include the motion of a slender ship along the center of a constant width rectangular channel by Tuck [2] and the lateral flow past a slender ship by Newman [3]. Tuck [4, 5] also considered unsteady constant-depth solutions relating to ship motions.

Plotkin [6] made a first attempt at studying the effect of variable water depth in the shallow-water hydrodynamics regime by considering the steady flow past an anchored slender ship in the presence of a slender bump. Beck, Newman and Tuck [7] treated the steady flow of a slender ship moving down the center of a dredged channel.

In this paper, the unsteady potential flow due to a slender ship translating past a wavy wall in shallow water is analyzed.

2. Problem formulation

Two Cartesian coordinate systems are introduced as shown in Figure 1. The primed system is fixed in space, with z' measured upwards from the undisturbed position of the free surface. The ship is translating with constant speed U in the negative x' direction. In this system, the ship is described by

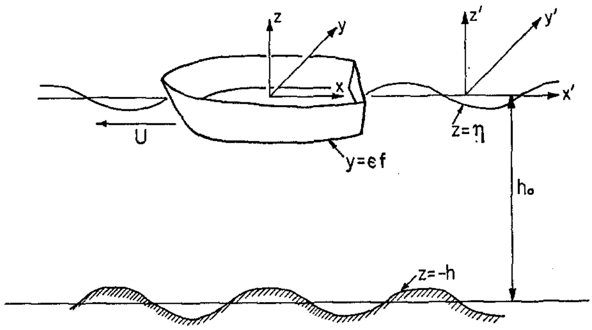

Figure 1. Coordinate system.

$$y' - \varepsilon f'(x', z', t') = 0. \tag{2.1}$$

The bottom is given by

$$z' + h'(x', y') = 0, \tag{2.2}$$

and the free surface is given by

$$z' - \eta'(x', y', t') = 0. \tag{2.3}$$

The flow is assumed to be irrotational which leads to the representation of the velocity as the positive gradient of a velocity potential $\phi'(x', y', z', t')$. Since the flow is also incompressible, the velocity potential must satisfy Laplace's equation

$$\phi'_{x'x'} + \phi'_{y'y'} + \phi'_{z'z'} = 0 \tag{2.4}$$

in the fluid domain.

The kinematic boundary condition on the ship surface for this inviscid flow requires that the body be a flow streamline or

$$\varepsilon f'_{t'} + \varepsilon\phi'_{x'}f'_{x'} + \varepsilon\phi'_{z'}f'_{z'} - \phi'_{y'} = 0, \text{ on } y' = \varepsilon f'. \tag{2.5}$$

The kinematic bottom condition is

$$\phi'_{z'} + \phi'_{x'}h'_{x'} + \phi'_{y'}h'_{y'} = 0, \text{ on } z' = -h'. \tag{2.6}$$

The free surface must also be a streamline of the flow or

$$\eta'_{t'} + \phi'_{x'}\eta'_{x'} + \phi'_{y'}\eta'_{y'} - \phi'_{z'} = 0, \text{ on } z' = \eta'. \tag{2.7}$$

The free surface also satisfies a dynamic boundary condition that the pressure is constant at its ambient value. Bernoulli's equation then takes the form

$$\phi'_{t'} + (\phi'^2_{x'} + \phi'^2_{y'} + \phi'^2_{z'})/2 + gz' = 0, \text{ on } z' = \eta', \tag{2.8}$$

where g is the gravitational acceleration.

The unprimed coordinate system moves with the ship, with the origin at midship. The two coordinate systems are related by the transformation

$$x' = x - Ut, \ y' = y, \ z' = z, \ t' = t. \tag{2.9}$$

In the unprimed system, the differential equation and boundary conditions become

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \text{ in fluid domain,} \tag{2.10}$$

$$\varepsilon U f_x + \varepsilon \phi_x f_x + \varepsilon \phi_z f_z - \phi_y = 0, \text{ on } y = \varepsilon f, \tag{2.11}$$

$$\phi_z + \phi_x h_x + \phi_y h_y = 0, \text{ on } z = -h, \tag{2.12}$$

$$\eta_t + U\eta_x + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0, \text{ on } z = \eta, \tag{2.13}$$

$$\phi_t + U\phi_x + (\phi_x^2 + \phi_y^2 + \phi_z^2)/2 + gz = 0, \text{ on } z = \eta. \tag{2.14}$$

It is noted that $f(x, y) = f'(x - Ut, y, t)$ is independent of time in this system and that the bottom description is now time dependent with $h(x, y, t) = h'(x - Ut, y)$.

The ship is slender which means that the beam and draft are small, $O(\varepsilon)$, with respect to the length $2l$. ε is the slenderness parameter and the hull description in equation (2.1) reflects the ordering. In the region near the ship, the inner region, the approximations of slender-ship theory apply. For the approximations of shallow-water theory to apply in the region far from the ship, the outer region, it is assumed that the water depth is of the same order as the draft, or $h = O(\varepsilon)$. Also, the Froude number based on depth, $F = U/(gh)^{\frac{1}{2}}$, is $O(1)$. This implies that the conventional Froude number based on l is small, $O(\varepsilon^{\frac{1}{2}})$.

The method of matched asymptotic expansions is used to define the mathematical problems in the inner and outer regions following Tuck [1].

3. The outer expansion

The outer region, far from the ship, is defined by the following order of magnitude of the coordinates with respect to ship length.

$$x, y = O(1), \; z = O(\varepsilon). \tag{3.1}$$

It is assumed that the velocity potential can be expanded in an asymptotic series in ε of the form

$$\phi = \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + \varepsilon^4\phi^{(4)} + \ldots. \tag{3.2}$$

Introduce the outer variable

$$Z = z/\varepsilon, \tag{3.3}$$

and denote the two-dimensional Laplacian by

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{3.4}$$

Substitution of the above into Laplace's equation (2.10) yields

$$\phi_{ZZ}^{(1)} = 0, \; \phi_{ZZ}^{(2)} = 0, \; \phi_{ZZ}^{(3)} = -\nabla^2\phi^{(1)}, \; \phi_{ZZ}^{(4)} = -\nabla^2\phi^{(2)}. \tag{3.5}$$

The bottom boundary condition (2.12) becomes

$$\phi_Z^{(1)}(-h) = 0, \; \phi_Z^{(2)}(-h) = 0, \; \phi_Z^{(3)}(-h) = -h_x\phi_x^{(1)}/\varepsilon - h_y\phi_y^{(1)}/\varepsilon,$$
$$\phi_Z^{(4)}(-h) = -h_x\phi_x^{(2)}/\varepsilon - h_y\phi_y^{(2)}/\varepsilon. \tag{3.6}$$

Equations (3.5) are integrated once with respect to Z and using equations (3.6), we get

$$\phi^{(1)} = \phi^{(1)}(x, y, t),\ \phi^{(2)} = \phi^{(2)}(x, y, t),$$
$$\phi_Z^{(3)} = -Z\nabla^2\phi^{(1)} - h\nabla^2\phi^{(1)}/\varepsilon - h_x\phi_x^{(1)}/\varepsilon - h_y\phi_y^{(1)}/\varepsilon,$$
$$\phi_Z^{(4)} = -Z\nabla^2\phi^{(2)} - h\nabla^2\phi^{(2)}/\varepsilon - h_x\phi_x^{(2)}/\varepsilon - h_y\phi_y^{(2)}/\varepsilon. \tag{3.7}$$

Substitute equations (3.7) into the dynamic free-surface condition (2.14) to get

$$-g\eta = \varepsilon\phi_t^{(1)} + \varepsilon U\phi_x^{(1)} + \varepsilon^2\phi_t^{(2)} + \varepsilon^2 U\phi_x^{(2)} + \varepsilon^2[\phi_x^{(1)2} + \phi_y^{(1)2}]/2 + O(\varepsilon^3). \tag{3.8}$$

Since $F = O(1)$, $U^2/g = O(\varepsilon)$ and therefore $\eta = O(\varepsilon^2)$. Let

$$\eta = \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + \dots, \tag{3.9}$$

and from (3.8),

$$\eta^{(2)} = -[\phi_t^{(1)} + U\phi_x^{(1)}]/g\varepsilon,$$
$$\eta^{(3)} = -[\phi_t^{(2)} + U\phi_x^{(2)} + (\phi_x^{(1)2} + \phi_y^{(1)2})/2]/g\varepsilon. \tag{3.10}$$

If equations (3.7) and (3.10) are now substituted into the kinematic free-surface condition (2.13), the differential equations for $\phi^{(1)}$ and $\phi^{(2)}$ are

$$gh\nabla^2\phi^{(1)} + gh_x\phi_x^{(1)} + gh_y\phi_y^{(1)} - \phi_{tt}^{(1)} - 2U\phi_{xt}^{(1)} - U^2\phi_{xx}^{(1)} = 0$$

and

$$gh\nabla^2\phi^{(2)} + gh_x\phi_x^{(2)} + gh_y\phi_y^{(2)} - \phi_{tt}^{(2)} - 2U\phi_{xt}^{(2)} - U^2\phi_{xx}^{(2)}$$
$$= [\phi_t^{(1)} + U\phi_x^{(1)}]\nabla^2\phi^{(1)} + 2[\phi_x^{(1)}\phi_{xt}^{(1)} + \phi_y^{(1)}\phi_{yt}^{(1)} + U\phi_x^{(1)}\phi_{xx}^{(1)} + U\phi_y^{(1)}\phi_{xy}^{(1)}]. \tag{3.11}$$

The preceding derivation follows very closely the analogous treatment of the constant-depth case by Tuck [1].

It is now assumed that the bottom location varies from its mean position by an amount of $O(\delta)$, where δ is a small parameter:

$$h' = h_0 + \delta h_1'(x', y').$$

The bottom variation is taken to be in the shape of a wavy wall of wave number k and if the frequency $\omega = kU$ is introduced, the description of the bottom in the ship-fixed system is

$$h = h_0[1 + \delta a \cos\omega(x/U - t)] = h_0 + \delta h_1(x, t). \tag{3.12}$$

The velocity potentials $\phi^{(1)}$ and $\phi^{(2)}$ are now expanded in series in δ

$$\phi^{(1)} = \phi^{(11)} + \delta\phi^{(12)} + O(\delta^2),$$

and

$$\phi^{(2)} = \phi^{(21)} + \delta\phi^{(22)} + O(\delta^2). \tag{3.13}$$

and therefore the outer velocity potential is

$$\phi = \varepsilon\phi^{(11)} + \varepsilon\delta\phi^{(12)} + \varepsilon^2\phi^{(21)} + O(\varepsilon^2\delta, \varepsilon\delta^2). \tag{3.14}$$

In this study, the terms $\phi^{(11)}$ and $\phi^{(12)}$ will be obtained which will give us the largest correction to the constant-depth result due to variable water depth. It is noted that the contribution to the forces and moments from the term $\phi^{(21)}$ may be comparable to those from $\phi^{(12)}$ but that this contribution is independent of depth variation and is obtained formally by Tuck [1].

Equations (3.12 and 3.14) are substituted into equation (3.10) to yield the following equations for $\phi^{(11)}$ and $\phi^{(12)}$

$$(1 - F_0^2)\phi_{xx}^{(11)} + \phi_{yy}^{(11)} = 0 \tag{3.15}$$

and

$$(1 - F_0^2)\phi_{xx}^{(12)} + \phi_{yy}^{(12)} - \phi_{tt}^{(12)}/gh_0 - 2U\phi_{xt}^{(12)}/gh_0 = -h_{1_x}\phi_x^{(11)}/h_0 - h_1 F_0^2\phi_{xx}^{(11)}/h_0, \tag{3.16}$$

where $F_0^2 = U^2/gh_0$, the Froude number based on mean depth.

To an observer in the outer region, as $\varepsilon \to 0$ the beam of the ship vanishes while the draft remains finite so that the ship appears to have collapsed onto the plane $y = 0$. We therefore seek solutions to equations (3.15–3.16) which are analytic everywhere with the possible exception of $y = 0$. It is noted that these equations are similar to those appearing in linearized subsonic aerodynamics and they will be solved using Green's function (source) distributions. The solution to equation (3.15) is

$$\phi^{(11)} = \int_{-\infty}^{\infty} G^{(11)}(x - \xi, y)A^{(11)}(\xi)d\xi \tag{3.17}$$

where

$$G^{(11)}(x, y) = (2\pi\beta)^{-1}\log(x^2 + \beta^2 y^2)^{\frac{1}{2}} \tag{3.18}$$

is the unit source potential and $\beta^2 = 1 - F_0^2$. $A^{(11)}(x)$ is the still unknown source strength.

To solve equation (3.16) it is convenient to use complex variables. Let

$$\phi^{(12)}(x, y, t) = \mathscr{R}[\bar{\phi}^{(12)}(x, y)e^{-i\omega t}], \tag{3.19}$$

and note that $h_1 = h_0 a\mathscr{R}[e^{i\omega x/U}e^{-i\omega t}]$. Upon substitution of these results into (3.16), the following equation for $\bar{\phi}^{(12)}$ is obtained:

$$\beta^2\bar{\phi}_{xx}^{(12)} + \bar{\phi}_{yy}^{(12)} + \omega^2\bar{\phi}^{(12)}/c^2 + 2U\omega i\bar{\phi}_x^{(12)}/c^2 = -ae^{i\omega x/U}[i\omega\phi_x^{(11)}/U + F_0^2\phi_{xx}^{(11)}] \tag{3.20}$$

where $c^2 = gh_0$, the square of the mean-depth shallow-water wave speed. The Green's function source potential for this equation is given in Robinson and Laurmann [8] as

$$\bar{G}^{(12)}(x, y) = i(4\beta)^{-1}e^{-i\omega F_0 x/c\beta^2}H_0^{(2)}[\omega(x^2 + \beta^2 y^2)^{\frac{1}{2}}/c\beta^2] \tag{3.21}$$

where $H_0^{(2)}$ is a Hankel function. The solution to equation (3.20) can now be written as

$$\bar{\phi}^{(12)} = \int_{-\infty}^{\infty} \bar{A}^{(12)}(\xi)\bar{G}^{(12)}(x - \xi, y)d\xi - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ae^{i\omega\xi/U}[i\omega\phi_x^{(11)}(\xi, \alpha)/U + F_0^2\phi_{xx}^{(11)}(\xi, \alpha)]\bar{G}^{(12)}(x - \xi, y - \alpha)d\xi\, d\alpha \tag{3.22}$$

where $\bar{A}^{(12)}(x)$ is the still unknown source strength.

4. The inner expansion

The inner region, in the neighborhood of the ship, is defined by the following order of magnitude of the coordinates with respect to the ship length

$$x = O(1),\ y,\ z = O(\varepsilon). \tag{4.1}$$

It is assumed that the velocity potential can be expanded in an asymptotic series in ε of the form

$$\phi = \varepsilon\Phi^{(1)} + \varepsilon^2\Phi^{(2)} + \ldots . \tag{4.2}$$

The inner variables Y and Z are defined as

$$Y = y/\varepsilon,\ \ Z = z/\varepsilon. \tag{4.3}$$

Equations (4.2–4.3) are substituted into Laplace's equation to yield

$$\Phi^{(1)}_{YY} + \Phi^{(1)}_{ZZ} = 0,\ \Phi^{(2)}_{YY} + \Phi^{(2)}_{ZZ} = 0. \tag{4.4}$$

It is seen that both $\Phi^{(1)}$ and $\Phi^{(2)}$ satisfy a two-dimensional Laplace equation in the cross-flow plane.

Conventional slender-body theory yields the hull boundary conditions

$$\Phi^{(1)}_N = 0,\ \Phi^{(2)}_N = Uf_x/(1 + f_Z^2)^{\frac{1}{2}},\ \text{on}\ Y = f, \tag{4.5}$$

where N is the normal in the cross-flow plane expressed in inner variables. There has been considerable discussion concerning the appropriate free-surface condition. Tuck [4, 5] uses the rigid-wall condition. Ogilvie and Tuck [9], in their consideration of ship motions in deep water, generate a free-surface condition which allows for waves in the inner region by allowing the frequency to become large. It is assumed here that the wave length of the bottom variation is of the order of the ship length or $\omega L/U = O(1)$. The ordering in the inner region then requires that the rigid-wall condition be satisfied

$$\Phi^{(1)}_Z = 0,\ \Phi^{(2)}_Z = 0,\ \text{on}\ \ Z = 0. \tag{4.6}$$

To obtain the boundary condition on the bottom, first substitute equations (4.2–4.3) into equation (2.12). Then introduce the bottom description (3.12) and expand the resulting equation about the mean position $Z = -h_0/\varepsilon$. Keeping terms linear in δ, we get

$$\Phi^{(1)}_Z - \delta h_1\Phi^{(1)}_{ZZ}/\varepsilon = 0,\ \Phi^{(2)}_Z - \delta h_1\Phi^{(2)}_{ZZ}/\varepsilon = 0,\ \text{on}\ Z = -h_0/\varepsilon. \tag{4.7}$$

Now, let

$$\Phi^{(1)} = \Phi^{(11)} + \delta\Phi^{(12)} + O(\delta^2). \tag{4.8}$$

$\Phi^{(11)}$ and $\Phi^{(12)}$ each satisfy Laplace's equation in the $Y-Z$ plane with zero normal derivatives on all boundaries in this plane. Therefore,

$$\Phi^{(1)} = \Phi^{(11)}(x) + \delta\Phi^{(12)}(x, t). \tag{4.9}$$

Now consider the problem for $\Phi^{(2)}$. The solution can be written as

$$\Phi^{(2)} = f(x, t) + \Phi^{(21)}(Y, Z, t; x) + \delta\Phi^{(22)}(Y, Z, t; x) \tag{4.10}$$

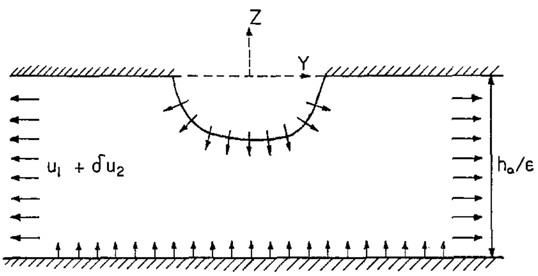

Figure 2. Inner region flowfield.

where $f(x, t)$ is arbitrary. $\Phi^{(21)}$ and $\Phi^{(22)}$ are uniquely defined if suitable boundary conditions at infinity are specified. Figure 2 shows a schematic of the inner region flowfield. For a hull symmetric with respect to the plane $Y = 0$, it seems reasonable to assume that

$$\Phi^{(21)} \to u_1(x)|Y| + o(1),\ \Phi^{(22)} \to u_2(x, t)|Y| + o(1) \tag{4.11}$$

as $|Y| \to \infty$ where u_1 and u_2 are determinable from conservation of mass.

The volume flux leaving the hull at x is

$$\int \Phi_N^{(21)} ds = US_x(x)$$

where the integral is taken around the wetted hull cross-section and $\varepsilon^2 S(x)$ is the area of that section below the plane $Z = 0$ ([1]). The volume flux leaving the bottom is

$$\delta \int_{-\infty}^{\infty} \Phi_Z^{(22)} dY = -\delta h_1 \int_{-\infty}^{\infty} \Phi_{YY}^{(21)} dY/\varepsilon = -2\delta h_1 u_1/\varepsilon.$$

Since half of the volume flux is channeled in each direction as $Y \to \pm\infty$, we have

$$u_1 = U\varepsilon S_x(x)/2h_0,\ u_2 = -U\varepsilon S_x(x)h_1/2h_0^2. \tag{4.12}$$

5. Matching

To determine the unknown functions $A^{(11)}(x)$, $\bar{A}^{(12)}(x)$, $\Phi^{(11)}(x)$ and $\Phi^{(12)}(x, t)$ the inner and outer expansions must be matched. The following matching principle from Van Dyke [10] is used:

„The m-term inner expansion of the (n-term outer expansion) =
the n-term outer expansion of the (m-term inner expansion) (5.1)

Take $m = 2$ and $n = 1$. The one-term outer expansion is

$$\varepsilon\phi^{(11)} + \varepsilon\delta\mathscr{R}[\bar{\phi}^{(12)} e^{-i\omega t}].$$

It has a two-term inner expansion of

$$\varepsilon\phi^{(11)}(x, o) + \varepsilon|y|\phi_y^{(11)}(x, o+) + \varepsilon\delta\mathscr{R}e^{-i\omega t}[\bar{\phi}^{(12)}(x, o) + |y|\bar{\phi}_y^{(12)}(x, o+)]$$
$$= \varepsilon\phi^{(11)}(x, o) + \varepsilon|y|A^{(11)}(x)/2 + \varepsilon\delta\mathscr{R}e^{-i\omega t}[\bar{\phi}^{(12)}(x, o) + |y|\bar{A}^{(12)}(x)/2]. \tag{5.2}$$

The two-term inner expansion is

$$\varepsilon\Phi^{(11)} + \varepsilon\delta\Phi^{(12)} + \varepsilon^2 f + \varepsilon^2\Phi^{(21)} + \varepsilon^2\delta\Phi^{(22)}.$$

It has a one-term expansion of

$$\varepsilon\Phi^{(11)} + \varepsilon\delta\Phi^{(12)} + \varepsilon|y|u_1 + \varepsilon\delta|y|u_2. \tag{5.3}$$

By equating equation (5.3) to equation (5.2) and using equation (4.12), the results of the matching are obtained as

$$\Phi^{(11)} = \phi^{(11)}(x, o),\ \Phi^{(12)} = \mathscr{R}[\mathrm{e}^{-i\omega t}\bar{\phi}^{(12)}(x, o)]$$
$$A^{(11)} = U\varepsilon S_x(x)h_0^{-1},\ \bar{A}^{(12)} = -Ua\varepsilon h_0^{-1}S_x(x)\mathrm{e}^{i\omega x/U} \tag{5.4}$$

The first-order velocity potential in the inner region is

$$\begin{aligned}\phi\varepsilon^{-1} &= U\varepsilon(2\pi\beta h_0)^{-1}\int_{-l}^{l} S_x(\xi)\log|x-\xi|\,d\xi \\ &- a\delta(4\beta)^{-1}\mathscr{R}i\,\mathrm{e}^{-i\omega t}\mathrm{e}^{-i\omega F_0 x/c\beta^2} \\ &\times\left\{U\varepsilon h_0^{-1}\int_{-l}^{l}\mathrm{e}^{i\omega\xi/cF_0\beta^2}S_x(\xi)H_0^{(2)}(\omega|x-\xi|/c\beta^2)\,d\xi\right. \\ &+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{i\omega\xi/cF_0\beta^2}[i\omega(F_0c)^{-1}\phi_x^{(11)}(\xi,\alpha) + F_0^2\phi_{xx}^{(11)}(\xi,\alpha)] \\ &\left.\times H_0^{(2)}[\omega((x-\xi)^2+\alpha^2\beta^2)^{\frac{1}{2}}/c\beta^2]\,d\xi\,d\alpha\right\}\end{aligned} \tag{5.5}$$

where

$$\phi^{(11)}(\xi,\alpha) = U\varepsilon(2\pi\beta h_0)^{-1}\int_{-l}^{l} S_x(\xi_0)\log[(\xi-\xi_0)^2+\beta^2\alpha^2]^{\frac{1}{2}}\,d\xi_0.$$

6. Inner expansion of pressure and forces

To first order in ε, the hydrodynamic pressure is obtained from the linearized Bernoulli equation as

$$p = -\rho(\phi_t + U\phi_x) \tag{6.1}$$

where ρ is the fluid density and ϕ is given in equation (5.5). Using equations (3.17–3.22, 4.8–4.9, 5.4), the pressure is

$$p = -\rho U\varepsilon\phi_x^{(11)}(x, o) - \rho\varepsilon\delta\mathscr{R}\mathrm{e}^{-i\omega t}[-i\omega\bar{\phi}^{(12)}(x, o) + U\bar{\phi}_x^{(12)}(x, o)]. \tag{6.2}$$

As in the constant-depth case [1], to this order the pressure is a function of x and t. It measures only the interaction between cross-sections as is evidenced by the integral representation of $\phi^{(11)}$ and $\bar{\phi}^{(12)}$. There is no dependence on the cross-section shape—the only hull geometry needed is the slope of the cross-sectional area curve $S_x(x)$.

The vertical force, positive upwards, is given by Tuck [1] as

$$L = \int_{-l}^{l} p(x)B(x)\,dx \tag{6.3}$$

where $B(x)$ is the width of the cross-section at the waterline. An integration by parts yields

$$L = \rho U\varepsilon \int_{-l}^{l} \phi^{(11)}(x, o)B_x(x)\,dx$$
$$+ \rho\varepsilon\delta\mathscr{R}\mathrm{e}^{-i\omega t} \int_{-l}^{l} [i\omega B(x) + UB_x(x)]\bar{\phi}^{(12)}(x, o)\,dx. \tag{6.4}$$

The first term is the mean-depth contribution to the force and is

$$L|_{\delta=0} = \rho U^2\varepsilon^2(2\pi\beta h_0)^{-1} \int_{-l}^{l}\int_{-l}^{l} S_x(\xi)B_x(x)\log|x-\xi|\,d\xi\,dx \tag{6.5}$$

which has been obtained by Tuck [1].

The trim moment about the y-axis, positive clockwise, is

$$M = -\int_{-l}^{l} xp(x)B(x)\,dx. \tag{6.6}$$

The counterpart of equation (6.4) is

$$M = -\rho U\varepsilon \int_{-l}^{l} \phi^{(11)}(x, o)(xB)_x\,dx$$
$$-\rho\varepsilon\delta\mathscr{R}\mathrm{e}^{-i\omega t} \int_{-l}^{l} [i\omega xB + U(Bx)_x]\bar{\phi}^{(12)}(x, o)\,dx. \tag{6.7}$$

The mean-depth contribution to the moment is

$$M|_{\delta=0} = -\rho U^2\varepsilon^2(2\pi\beta h_0)^{-1} \int_{-l}^{l}\int_{-l}^{l} S_x(\xi)(xB)_x\log|x-\xi|\,d\xi\,dx. \tag{6.8}$$

Non-dimensional force and moment coefficients can be defined as

$$C_L = \frac{L}{\frac{1}{2}\rho U^2(2l)^2}, \quad C_M = \frac{M}{\frac{1}{2}\rho U^2(2l)^3}, \tag{6.9}$$

7. Sample problem

Consider a hull of revolution with a parabolic waterline. The width and cross-sectional area are given by

$$B(x) = 2\varepsilon B_0(1 - x^2/l^2)$$

and

$$\varepsilon^2 S(x) = \pi B^2/8. \tag{7.1}$$

The mean-depth contributions to the force and moment, equations (6.5 and 6.8), are

$$L|_{\delta=0} = -16\rho U^2\varepsilon^3 B_0^3(9\beta h_0)^{-1}, \quad M|_{\delta=0} = 0. \tag{7.2}$$

The unsteady contributions to the force and moment may now be calculated from equations

(6.4 and 6.7). Using equation (6.9), we write

$$C_L = -8\varepsilon^3 B_0^3(9\beta h_0 l^2)^{-1} + \mathscr{R}[e^{-i\omega t}R_L e^{i\theta_L}]$$

and

$$C_M = \mathscr{R}[e^{-i\omega t}R_M e^{i\theta_M}] \tag{7.3}$$

where the unsteady force and moment coefficients are written in exponential form. These coefficients depend basically on two non-dimensional parameters; the Froude number F_0, and the reduced frequency, $\omega l/c$, which is equal to π times the ratio of ship length to wave length. The coefficients are obtained by numerical integration using the Gaussian quadrature formulas in Stroud and Secrest [11]. Legendre polynomials are used for the streamwise integration and Laguerre polynomials for the transverse integration.

The theory requires that $F_0 = 0(1)$ and only subcritical flow is considered. The theory becomes invalid as the Froude number approaches one since the ordering of terms in the outer-region equations must be reconsidered. To study the effect of Froude number variation, the coefficients have been calculated for $F_0 = .5, .7$ and $.9$. To insure that changes in the streamwise direction are of the order of the ship length so that slender-body theory is applicable in the inner region, it has been assumed that $\omega l/c = 0(1)$. The theory should then apply in the low-frequency limit but should be invalid as $\omega l/c \to \infty$.

Consider the limiting case $\omega l/c \to 0$. In this limit h_1 becomes constant and therefore $h = h_0 + \delta h_1$ is also constant. Equation (3.10) reduces to

$$(1 - F^2)\phi_{xx}^{(1)} + \phi_{yy}^{(1)} = 0 \tag{7.4}$$

and the solutions for the force and moment are obtained by replacing F_0 by F in equations (6.4 and 6.8), respectively. An expansion of these results for small δ yields

$$\begin{matrix} L|_{\omega l/c=0} \\ M|_{\omega l/c=0} \end{matrix} = \left\{1 - \frac{\delta h_1}{h_0}\left[1 + \frac{1}{2}\,\frac{F_0^2}{1 - F_0^2}\right]\right\}\begin{matrix} L|_{\delta=0} \\ M|_{\delta=0} \end{matrix} \tag{7.5}$$

The solution technique of this paper, developed for a time-dependent h_1, linearizes equation (3.10) so that the δ dependence is contained in equation (3.16).

The unsteady force coefficient magnitude R_L is presented as a function of reduced frequency $\omega l/c$ in Figure 3. The value of R_L from equation (7.5), the "exact" first order limit for $\omega l/c \to 0$, is also shown. The agreement at this limit is excellent. The curves all exhibit peaks in the low-frequency range with the magnitude of the peaks increasing monotonically with Froude number. It is interesting to note that the peaks occur at rather large values of the ratio of wave length to ship length, greater than 2π for the cases considered. The unsteady force coefficient argument is shown in Figure 4. In the limit as $\omega l/c \to 0$, $\theta_L \to 0$.

The unsteady moment coefficient magnitude R_M is displayed in Figure 5. It is seen that the curves resemble those for the force coefficient with the peaks shifted somewhat to higher values of frequency. The correct limit $R_M = 0$ is approached for $\omega l/c = 0$. The unsteady moment coefficient argument θ_M is shown in Figure 6. In the limit $\omega l/c \to 0$, it is noted that $\theta_M \to \pi/2$.

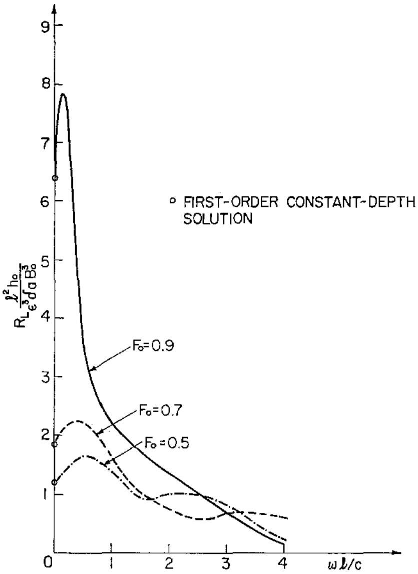

Figure 3

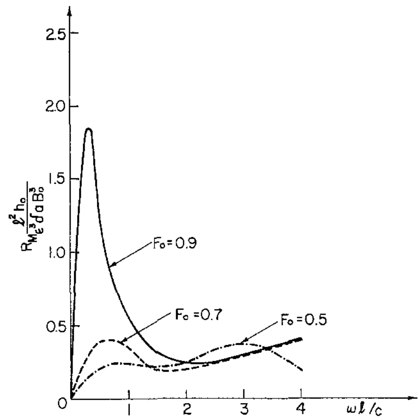

Figure 5

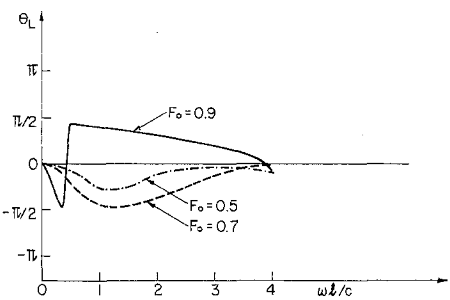

Figure 4

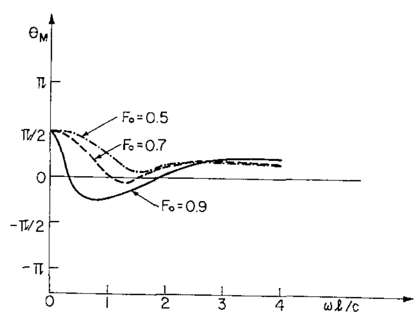

Figure 6

Figure 3. Magnitude of unsteady vertical force coefficient for ship with parabolic waterline of beam $2\varepsilon B_0$, half-length l, in water of mean depth h_0, plotted against reduced frequency $\omega l/c = \pi$ (ship length/wave length).
Figure 4. Argument of unsteady vertical force coefficient under the same conditions as Figure 3.
Figure 5. Magnitude of unsteady pitching moment coefficient under the same conditions as Figure 3.
Figure 6. Argument of unsteady pitching moment coefficient under the same conditions as Figure 3.

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